Announcements

1) Advising Day for math majors 3/19 (Monday) 2-3(?), pizza

2) Midterms - take-home midterm up on CTools, due Thursday 3/15. In-class exam Tuesday 3/13
(definitions, examples, statements of theorems, one proof)

Recall: Alternating Series Test

- proof on the notes for last class

Statement: Suppose $(b_n)_{n\in\mathbb{N}}$ satisfies $b_n \geq 0$ $\forall\, n \in \mathbb{N}$ and

1) $b_n \geq b_{n+1}$ $\forall\, n \in \mathbb{N}$

2) $\lim\limits_{n\to\infty} b_n = 0$.

Then $\sum\limits_{n=1}^{\infty} (-1)^n b_n$ converges

Example 1: We can now manufacture uncountably many conditionally convergent series using the alternating series test.

For one, let $b_n = \frac{1}{n}$.

Then $b_n \geq 0$, and

$$b_n = \frac{1}{n} \geq \frac{1}{n+1} = b_{n+1}$$

since $n+1 \geq n$. Finally,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

By the alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} \quad \text{converges.}$$

We already know that

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges,}$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally

convergent.

Similarly, if $b_n = \frac{1}{n^p}$

for $\boxed{0 < p \leq 1}$, then

$(b_n)_{n=1}^{\infty}$ satisfies the hypotheses of the alternating series test, so

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^p}$ converges. But

by Cauchy condensation,

$$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges}$$

Fun with conditional convergence

Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection.

Starting with $\sum_{n=1}^{\infty} a_n$, the

series $\sum_{n=1}^{\infty} a_{\varphi(n)}$ is called a

rearrangement of $\sum_{n=1}^{\infty} a_n$.

Example 2: Define $\varphi : \mathbb{N} \to \mathbb{N}$

by
$$\varphi(2k) = 2k-1 \quad \forall\, k \in \mathbb{N}$$
and
$$\varphi(2k-1) = 2k \quad \forall\, k \in \mathbb{N}$$

Then φ sends even numbers to odd numbers and odd numbers to even numbers.

Let $a_n = \dfrac{(-1)^n}{n}$.

$$\sum_{n=1}^{\infty} a_n = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \text{———}$$

$$\sum_{n=1}^{\infty} a_{\varphi(n)} = a_{\varphi(1)} + a_{\varphi(2)} + a_{\varphi(3)} + \cdots$$

$$= a_2 + a_1 + a_4 + \cdots$$

$$= \frac{1}{2} - 1 + \frac{1}{4} - \frac{1}{3} + \cdots$$

We know $\sum_{n=1}^{\infty} a_n$ converges

by alternating series test.

Does $\sum_{n=1}^{\infty} a_{\varphi(n)}$ even converge?

If so, to what?

You can't use the alternating series test to show convergence since it is not always true that

$$|a_{\varphi(n)}| \geq |a_{\varphi(n+1)}|.$$

What do you do?

Extra credit: answer these questions!

Lemma: (Exercise 2.7.3)

Suppose $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Then if

$$b_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$c_n = \begin{cases} a_n, & \text{if } a_n \leq 0 \\ 0, & \text{if } a_n > 0, \end{cases}$$

both series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ diverge

Proof: Take the contrapositive.

Suppose one of $\sum_{n=1}^{\infty} b_n$ or $\sum_{n=1}^{\infty} c_n$ converges (or maybe both converge). Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(This assumes $\sum_{n=1}^{\infty} a_n$ converges.)

Assume, without loss of generality, that $\sum_{n=1}^{\infty} b_n$ converges.

Observe that

$$a_n = b_n + c_n \quad \forall\, n \in \mathbb{N}$$

Hence

$$a_n - b_n = c_n, \quad \text{so}$$

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

and $\sum_{n=1}^{\infty} c_n$ converges, Now $|a_n| = b_n - c_n$, so

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (b_n - c_n)$$

$$= \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} c_n$$

Since $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ converge

This shows $\sum_{n=1}^{\infty} |a_n|$ converges,

so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\square$

Theorem. Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Pick $x \in \mathbb{R}$. Then $\exists$ a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to x!

proof: Since $\sum_{n=1}^{\infty} a_n$ converges conditionally, if we let $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be as in the previous lemma,

$$\boxed{\sum_{n=1}^{\infty} b_n \text{ and } \sum_{n=1}^{\infty} c_n \text{ diverge,}}$$

Since both series diverge, it is implicit that $(a_n)_{n\in\mathbb{N}}$ has infinitely many positive terms and infinitely many negative terms.

Assume for simplicity that

$a_n \neq 0 \ \forall n.$

Warning! What follows will most likely confuse more than enlighten

Now $\mathbb{N} = N_1 \sqcup N_2$ where

$N_1 = \{ n \in \mathbb{N} \mid b_n \neq 0 \}$

$N_2 = \{ n \in \mathbb{N} \mid c_n \neq 0 \}$

As both these sets are infinite, $\exists$ bijections $\pi : \mathbb{N} \to N_1$, $\gamma : \mathbb{N} \to N_2$. Define $(p_n)_{n \in \mathbb{N}}$ by $p_n = a_{\pi(n)}$ and

$(q_n)_{n \in \mathbb{N}}$ by $q_n = a_{\gamma(n)}$.

Note that $\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} c_n$, so $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both diverge. For further simplicity, let $x \geq 0$, Since $\sum_{n=1}^{\infty} p_n$ diverges, $\exists$

$J_1 \in \mathbb{N}$ with

$$\sum_{n=1}^{J_1} p_n + q_1 > x.$$

Since $\sum_{n=1}^{\infty} q_n$ diverges, $\exists K_1 \in \mathbb{N}$,

$K_1 > 1$,

$$\sum_{n=1}^{J_1} p_n + \sum_{n=1}^{K_1 - 1} q_n > x > \sum_{n=1}^{J_1} p_n + \sum_{n=1}^{K_1} q_n$$

Again since $\sum_{n=1}^{\infty} p_n$ diverges,

$\exists \; J_2 > J_1$ with

$$\sum_{n=1}^{J_2-1} p_n + \sum_{n=1}^{K_1} q_n < x < \sum_{n=1}^{J_2} p_n + \sum_{n=1}^{K_1} q_n .$$

Completing the initial steps, since $\sum_{n=1}^{\infty} q_n$ diverges, $\exists \; K_2 > K_1$ with

$$\sum_{n=1}^{J_2} p_n + \sum_{n=1}^{K_2-1} q_n > x > \sum_{n=1}^{J_2} p_n + \sum_{n=1}^{K_2} q_n$$

Now suppose $K_1 < K_2 < \cdots < K_m$ and $J_1 < J_2 < \cdots < J_m$ are inductively chosen.

Pick $J_{m+1} > J_m$ with

$$\sum_{n=1}^{J_{m+1}-1} p_n + \sum_{n=1}^{K_m} q_n < x < \sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_m} q_n .$$

Then choose $K_{m+1} > K_m$ with

$$\sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}-1} q_n > x > \sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}} q_n$$

In this way, we define sequences $(J_m)_{m=1}^{\infty}$ and $(K_m)_{m=1}^{\infty}$ with

$$\sum_{n=1}^{J_{m+1}-1} p_n + \sum_{n=1}^{K_m} q_n < x < \sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_m} q_n .$$

and

$$\sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}-1} q_n > x > \sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}} q_n$$

$\forall\ m \geq 2$ (J_1, K_1 chosen as specified earlier)

Now define $\varphi : \mathbb{N} \to \mathbb{N}$ by

$$\varphi(n) = \begin{cases} \pi(n), & 1 \le n \le J_1, \\ & J_m + K_m < n \le K_m + J_{m+1} \\ & \text{for some } m \in \mathbb{N} \\ \gamma(n), & J_1 < n \le J_1 + K_1, \\ & J_{m+1} + K_m < n < J_{m+1} + K_{m+1} \\ & \text{for some } m \in \mathbb{N} \end{cases}$$

Then

$$\underbrace{\sum_{n=1}^{J_{m+1}-1} p_n + \sum_{n=1}^{K_m} q_n}_{=\sum_{n=1}^{K_m+J_{m+1}-1} a_{\varphi(n)}} < x < \underbrace{\sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_m} q_n}_{=\sum_{n=1}^{K_n+J_{m+1}} a_{\varphi(n)}}$$

and

$$\underbrace{\sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}-1} q_n}_{=\sum_{n=1}^{J_{m+1}+K_{m+1}-1} a_{\varphi(n)}} > x > \underbrace{\sum_{n=1}^{J_{m+1}} p_n + \sum_{n=1}^{K_{m+1}} q_n}_{=\sum_{n=1}^{J_{m+1}+K_{n+1}} a_{\varphi(n)}},$$

So from these inequalities,

$$\left(\sum_{n=1}^{K_m+J_{m+1}} a_{\varphi(n)}\right) - X < \sum_{n=1}^{K_m+J_{m+1}} a_{\varphi(n)} - \sum_{n=1}^{K_m+J_{m+1}-1} a_{\varphi(n)}$$

$$= a_{\varphi(K_m+J_{m+1})}$$

Now if $K_m + J_{m+1} \leq t < K_{m+1} + J_{m+1}$,

$$\left(\sum_{n=1}^{t} a_{\varphi(n)}\right) - X < \left(\sum_{n=1}^{K_m+J_{m+1}} a_{\varphi(n)}\right) - X$$

$$< a_{\varphi(K_m+J_{m+1})}$$

Then

$$X - \sum_{n=1}^{K_{m+1}+J_{m+1}} a_{\varphi(n)} \leq \sum_{n=1}^{K_{m+1}+J_{m+1}-1} a_{\varphi(n)} - \sum_{n=1}^{K_{m+1}+J_{m+1}} a_{\varphi(n)}$$

$$= a_{\varphi(K_{m+1}+J_{m+1})}$$

Now if $K_{m+1}+J_{m+1} \leq t < K_{m+1}+J_{m+2}$

$$X - \sum_{n=1}^{t} a_{\varphi(n)} \leq X - \sum_{n=1}^{K_{m+1}+J_{m+1}} a_{\varphi(n)}$$

$$= a_{\varphi(K_{m+1}+J_{m+1})}$$

We have just shown that

$$\left| x - \sum_{n=1}^{t} a_{\varphi(n)} \right|$$

$$< \begin{cases} a_{\varphi(K_m + J_{m+1})}, & K_m + J_{m+1} \le t < K_{m+1} + J_{m+1} \\ a_{\varphi(K_{m+1} + J_{m+1})}, & K_{m+1} + J_{m+1} \le t < K_{m+1} + J_{m+2} \end{cases}$$

Since $(K_m)_{m \in \mathbb{N}}$ and $(J_m)_{m \in \mathbb{N}}$ are increasing and $\sum_{n=1}^{\infty} a_n$ converges

$$\Rightarrow \lim_{n \to \infty} a_n = 0,$$

$\lim_{n \to \infty} a_{\varphi(n)} = 0$. Choose

$N_1 \in \mathbb{N}$ so that

$$|a_{\varphi(n)}| < \varepsilon \quad \forall \ n \geq N_1$$

Choose $M \in \mathbb{N}$ so that

$$N = k_M + J_{M+1} > N_1 .$$

Then $\forall \ t \geq N$,

$$\left| x - \sum_{n=1}^{t} a_{\varphi(n)} \right| < \varepsilon, \text{ so}$$

$$\sum_{n=1}^{\infty} a_{\varphi(n)} = x .$$

The proof for $x < 0$ merely interchanges the roles of $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ $\square$

In fact, the situation is even worse for conditionally convergent series!

Rudin: If $x \leq y$, then $\exists$ a rearrangement of $\sum_{n=1}^{\infty} a_n$ that satisfies

$$\liminf_{k \to \infty} \sum_{n=1}^{k} a_n = x, \quad \limsup_{k \to \infty} \sum_{n=1}^{k} a_n = y$$

Theorem: (ratio test) Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ $\forall n \in \mathbb{N}$. Then

a) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

b) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

c) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, the test is inconclusive.

Proof.

c) Let $a_n = \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges, but $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.

Let $a_n = \frac{1}{n}$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

a) Suppose $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$

Let $\varepsilon = \frac{L-1}{2}$ and choose

$N \in \mathbb{N}$ so that

$$\left| \frac{|a_{n+1}|}{|a_n|} - L \right| < \frac{L-1}{2} \quad \forall n \geq N.$$

Then

$$-\frac{L+1}{2} < \frac{|a_{n+1}|}{|a_n|} - L < \frac{L-1}{2}, \text{ so}$$

adding L,

$$\frac{L+1}{2} < \frac{|a_{n+1}|}{|a_n|} < \frac{3L-1}{2} < 1 \quad \forall n \geq N$$

$$\Rightarrow \boxed{|a_{n+1}| < \frac{3L-1}{2} |a_n|} \quad \forall n \geq N.$$

Now let $n > N$ and inductively assume that

$$|a_n| < \left(\frac{3L-1}{2}\right)^{n-N} |a_N|$$

Then $|a_{n+1}| < \left(\frac{3L-1}{2}\right) |a_n|$

$$< \left(\frac{3L-1}{2}\right)^{n+1-N} |a_N|.$$

We have then $\forall\ n \geq N$,

$$\boxed{|a_n| \leq \left(\frac{3L-1}{2}\right)^{n-N} |a_N|.}$$

Now

$$\sum_{n=N}^{\infty} \left(\frac{3L-1}{2}\right)^{n+1-N} |a_N|$$

is a geometric series, and since $\frac{3L-1}{2} < 1$, the series converges.

Therefore, by comparison,

$$\sum_{n=N}^{\infty} |a_n|$$ converges. This

implies $$\sum_{n=1}^{\infty} |a_n|$$ converges,

and so $$\sum_{n=1}^{\infty} a_n$$ converges absolutely

b) The proof is similar to a), except now choose N so that

$$\left| \frac{|a_{n+1}|}{|a_n|} - L \right| < \frac{L-1}{2}$$

$\forall\, n \geq N$ Then

$$\frac{1-L}{2} < \frac{|a_{n+1}|}{|a_n|} - L < \frac{L-1}{2}$$

so adding L,

$$\frac{1+L}{2} < \frac{|a_{n+1}|}{|a_n|} < \frac{3L-1}{2}.$$

This implies

$$\left(\frac{1+L}{2}\right)|a_n| < |a_{n+1}| \quad \forall\, n \geq N, \text{ so}$$

again via induction, we get

$$\left(\frac{1+L}{2}\right)^{n-N} |a_N| < |a_n| \quad \forall\, n \geq N.$$

Then $$\lim_{n\to\infty} |a_n| > \lim_{n\to\infty} \left(\frac{1+L}{2}\right)^{n-N} |a_N|$$

$$= \infty$$

since $\frac{L+1}{2} > 1$.

This in turn implies

$\lim_{n\to\infty} a_n \neq 0$, and so

$\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Example 3: (ratio test)

Theorem. (abs. conv. rearrangement)

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any rearrangement also converges to $\sum_{n=1}^{\infty} a_n$.

Corollary (products)